LAKE FOREST COLLEGE

Extracting π from Chaos

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Motivation

- Classical Definition: The ratio of a circle's circumference to its diameter (Euclidean geometry).
- Standard Computation include
  - Infinite series [3]
  - Geometric limits [1]
  - Probabilistic methods [2]

- Milton F. Maritz (2023) proposes extracting π not from geometry, but from the statistical properties of a *chaotic* dynamical system [4].

What is Chaos?

- Chaos occurs in *deterministic* nonlinear systems that exhibit extreme sensitivity to initial conditions.
- Although these systems follow strict rules, their long-term behavior is effectively unpredictable and mimics randomness.
- Common examples where chaos arises include
  - Atmospheric convection (Lorenz system)
  - Double pendulum
  - Population growth models
- We will show that hidden within the *noise* of a specific population model lies the geometry of π.

Representing a Chaotic Model

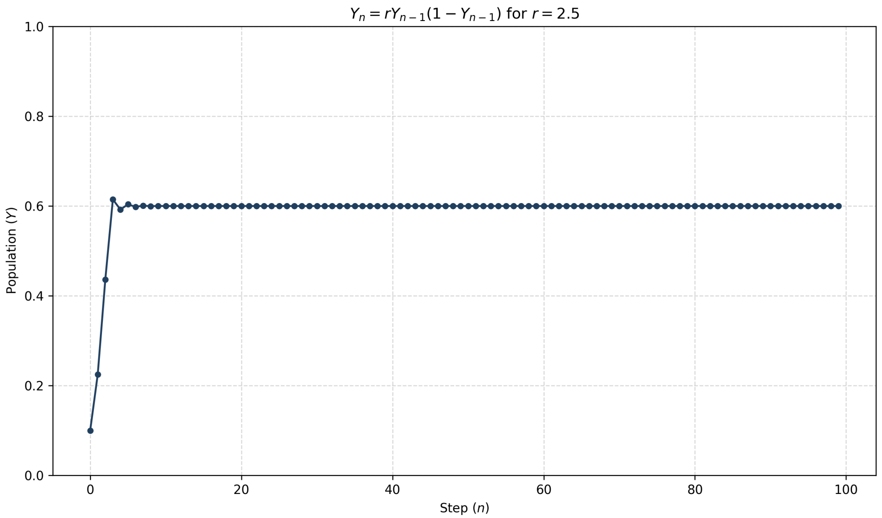
Population Model

The Logistic Map

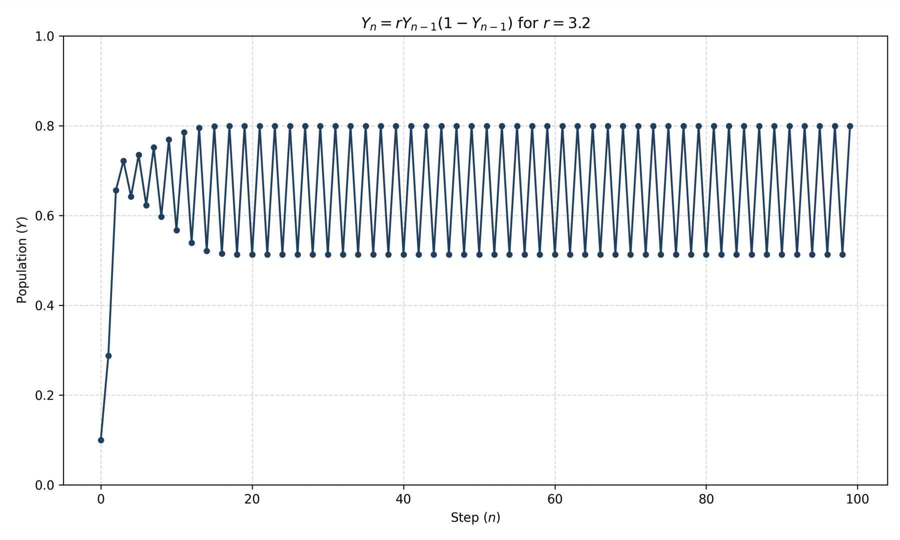
$$Y_{n+1} = rY_n(1 - Y_n)$$

is a classic model for population growth, where $Y_n \in [0, 1]$ represents the population density, and $r \in [0, 4]$ is the growth rate parameter.

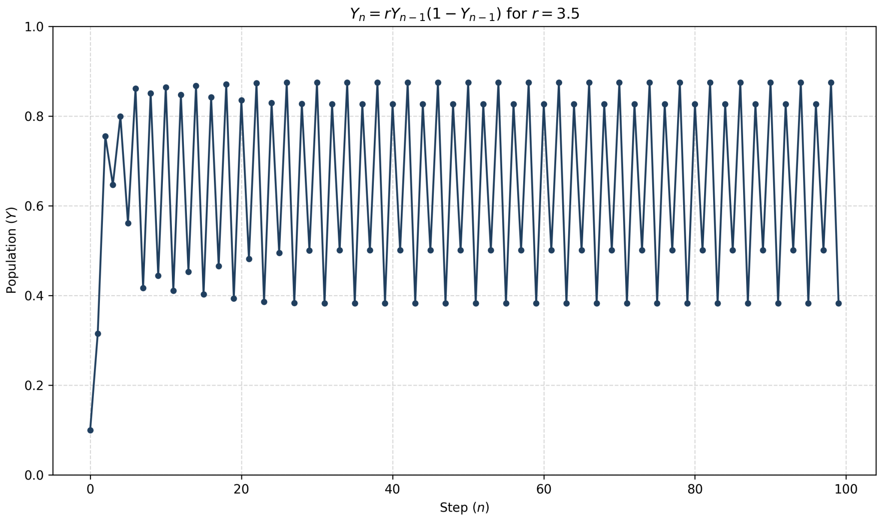
We simulate this model with a fixed initial condition $Y_0 = 0.1$ to observe how changing r dramatically alters behavior from stability to chaos.



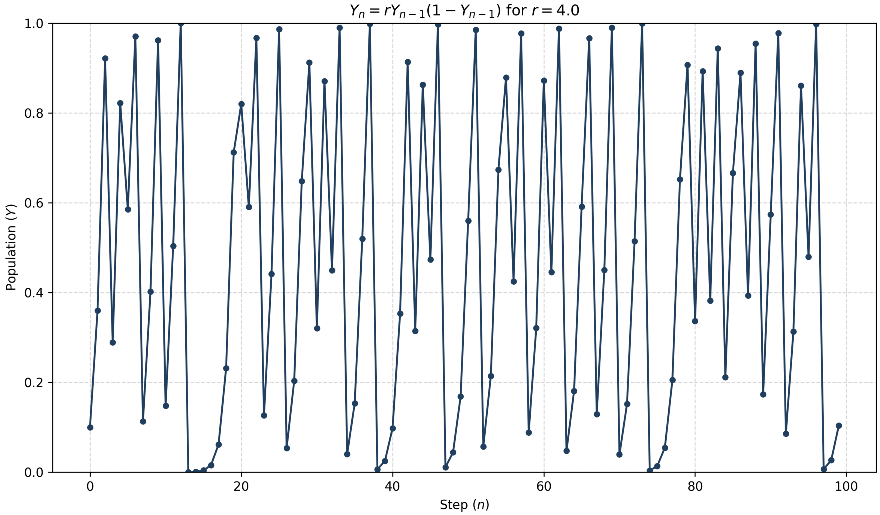
Stable behavior, with $r = 2.5$.



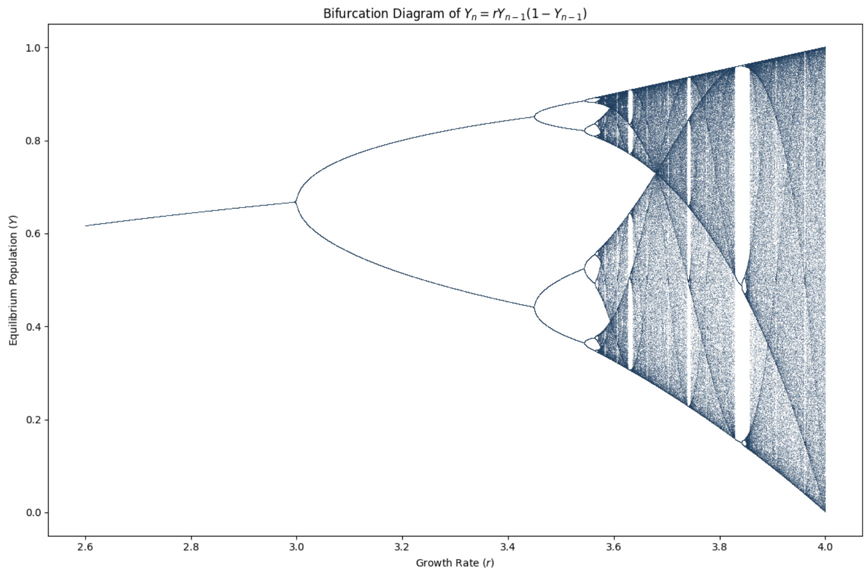
Periodic behavior, with $r = 3.2$.



Periodic behavior with higher period, with $r = 3.5$.



Complete Chaos! With $r = 4$.



As r increases, the system goes from stability ($r < 3$) to periodicity ($r < 3.6$) to chaos ($r \approx 4$).

Trigonometric Representation

We have found that at $r = 4$, the model $Y_{n+1} = 4Y_n(1 - Y_n)$ exhibits deterministic chaos. But how do we connect this to π?

We switch to a coordinate system governed by circles.

Consider the substitution $Y_n = \sin^2(\theta_n)$.

This is valid because $\sin^2(\theta) \in [0, 1]$, matches the domain of Y_n.

Substituting into the Logistic Map, we get

$$\sin^2(\theta_{n+1}) = 4\sin^2(\theta_n)(1 - \sin^2(\theta_n))$$

Recall the identities $1 - \sin^2(x) = \cos^2(x)$ and $\sin(2x) = 2\sin(x)\cos(x)$. Applying these to our equation, we get

$$\begin{aligned} \sin^2(\theta_{n+1}) &= 4\sin^2(\theta_n)(1 - \sin^2(\theta_n)) \\ &= 4\sin^2(\theta_n)\cos^2(\theta_n) \\ &= (2\sin(\theta_n)\cos(\theta_n))^2 \\ &= \sin^2(2\theta_n) \end{aligned}$$

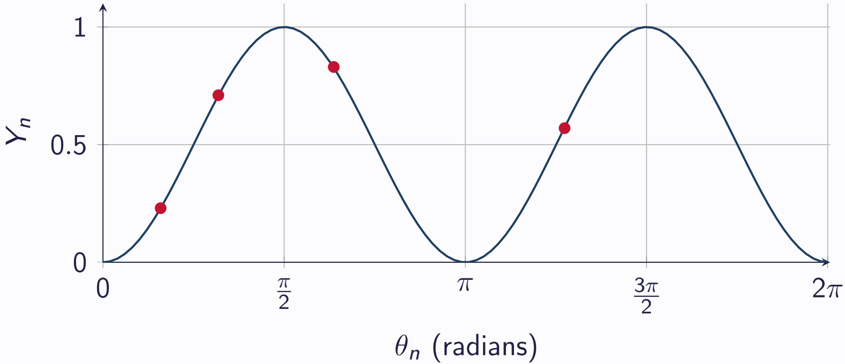
Hence, $\theta_{n+1} = 2\theta_n$ which implies

$$Y_n = \sin^2(2^n\theta_0) \quad \text{and} \quad \theta_n = 2^n\theta_0$$

The complex chaotic behavior of Y_n is actually just a simple doubling of the angle θ at every step.

Mapping Chaos to π

Since $\theta_n = 2^n\theta_0$, the angle wraps around the unit circle repeatedly. The *random* visits on $[0, 1]$ by Y_n are actually points sampled from the periodic curve $Y = \sin^2(\theta)$, which is governed by π.



Extracting π Statistically

Deriving the Probability Density

Assume the angle θ_n behaves like a random variable uniformly distributed over the interval $(0, \pi/2)$, then

$$\theta \sim \mathcal{U}(0, \pi/2) \implies \mathbb{P}_\Theta(\theta) = \frac{2}{\pi}$$

We want to find the probability density function (PDF) of the population values $Y = \sin^2(\theta)$. Using the change of variables formula, we have

$$\mathbb{P}_Y(y) = \mathbb{P}_\Theta(\theta) \left| \frac{d\theta}{dy} \right|$$

Inverting the transformation $y = \sin^2(\theta) = f(\theta)$ gives

$$\theta = f^{-1}(y) = \arcsin(\sqrt{y})$$

Deriving the Probability Density

Computing the derivative of the inverse transformation gives

$$\frac{d\theta}{dy} = \frac{d}{dy}\arcsin(\sqrt{y}) = \frac{1}{\sqrt{1-y}} \cdot \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y-y^2}}$$

Substituting this and $\mathbb{P}_\Theta(\theta) = \frac{2}{\pi}$ into the PDF formula, we get

$$\mathbb{P}_Y(y) = \frac{2}{\pi} \cdot \frac{1}{2\sqrt{y-y^2}} = \frac{1}{\pi\sqrt{y-y^2}}$$

Observe, this is a valid PDF for $y \in (0,1)$ since

$$\int_0^1 \frac{1}{\pi\sqrt{y-y^2}}\,dy = 1$$

(This is known as the arcsine distribution).

π from Histogram

To estimate π statistically, we generate N points using the logistic map

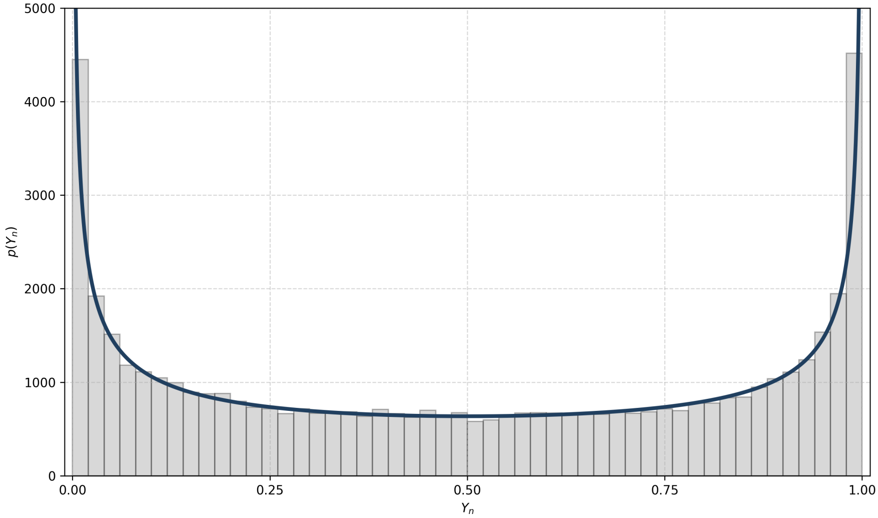
$$Y_{n+1} = 4Y_n(1 - Y_n)$$

with Y_0 chosen uniformly at random in $[0, 1]$, and the values are binned into B equally spaced bins over $[0, 1]$. We can compare this simulated histogram with a scaled version of the derived PDF, denoted as a likelihood function

$$L(y) = \left(\frac{N}{B}\right)\left(\frac{1}{\pi\sqrt{y - y^2}}\right)$$

where the scaling factor $\frac{N}{B}$ ensures that the area under the likelihood function matches the total number of samples per bin.

We run this experiment for $N = 50,000$ and $B = 50$ bins.



(The line represents the likelihood function $L(y)$)

π from Histogram

We can derive an estimate for π based on the bin count. Let j be the count of points in a specific bin centered at y, with width $\frac{1}{B}$.

The theoretical expected count for this bin is given by $L(y)$. Setting the observed count j equal to the theoretical count:

$$j \approx L(y) = \frac{N}{B\pi\sqrt{y - y^2}}$$

Solving for π, we get an estimator for each bin centered at y as

$$P_{\text{hist}}(y) = \frac{N}{B \cdot j\sqrt{y - y^2}} \approx \pi$$

π from Histogram

Some values of $P_{\text{hist}}(y) = \hat{\pi}$ from the histogram are

y	Count (j)	Estimate ($\hat{\pi}$)
0.3	669	3.23198
0.4	666	3.05287
0.5	581	3.44303
0.6	671	3.05549
0.7	669	3.29417

π from Single Bin

Instead of maintaining a full histogram, we can save memory by counting points in just one specific bin.

Assume we choose the bin centered at $y = 0.5$, with width $h = \frac{1}{B}$ (just for easy computation).

At $y = 0.5$, the term $\sqrt{y - y^2} = \sqrt{0.5 - 0.25} = 0.5$.

Substituting this into our histogram formula simplifies it beautifully to

$$P_{\text{bin}}(0.5) = \frac{N}{B \cdot j \cdot (0.5)} = \frac{2N}{B \cdot j} \approx \pi$$

π from Single Bin

Simulation gives,

N	B	Count (j)	Estimate ($\hat{\pi}$)	Error ($\pi - \hat{\pi}$)
$10,000$	40	165	3.03030	1.11×10^{-1}
$100,000$	100	656	3.04878	9.28×10^{-2}
$1,000,000$	100	$6,468$	3.09215	4.94×10^{-2}
$10,000,000$	100	$63,912$	3.12930	1.23×10^{-2}
$100,000,000$	100	$636,347$	3.14294	1.35×10^{-3}

As N increases, the approximation converges to π, confirming that we have successfully extracted the constant from the chaotic data.

Extracting π Geometrically

Reversing the Chaos

We have seen that iterating *forward* ($n \to n+1$) doubles the angle θ_n, causing chaos

$$\theta_{n+1} = 2\theta_n$$

What if we iterate *backward* ($n \to n-1$)?

Inverting the angular equation $\theta_{n+1} = 2\theta_n$ gives us

$$\theta_{n-1} = \frac{\theta_n}{2}$$

Backward iteration halves the angle. This suggests that the system should become more stable and predictable as we go back in time.

Backward Iteration

Recall the relationship between population Y and angle θ given by

$$Y_n = \sin^2(\theta_n)$$

We can invert the logistic map $Y_n = 4Y_{n-1}(1 - Y_{n-1}) \implies 4Y_{n-1}^2 - 4Y_{n-1} + Y_n = 0$ using the quadratic formula to find

$$Y_{n-1} = \frac{1 \pm \sqrt{1 - Y_n}}{2}$$

We choose the negative root to correspond to the principal half-angle $\theta/2$:

$$Y_{n-1} = \frac{1 - \sqrt{1 - Y_n}}{2}$$

Connection to π

Start at the peak of the sine curve, $Y_0 = 1$ (i.e., $\theta_0 = \pi/2$). Iterating backward n times, we get

$$\theta_{-n} = \frac{\pi/2}{2^n} = \frac{\pi}{2^{n+1}}$$

since each step halves the angle.

Observe that for a small angle θ, we get $\sin(\theta) \approx \theta$.

Thus, we can approximate Y_{-n} as

$$\sqrt{Y_{-n}} = \sin(\theta_{-n}) \approx \theta_{-n} = \frac{\pi}{2^{n+1}}$$

Vieta's Formula

Rearranging the approximation gives us an estimate for π as

$$\pi \approx 2^{n+1}\sqrt{Y_{-n}}$$

This recursive square root structure is equivalent to [5], one of the first infinite products derived for π, given by

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

This beautiful connection validates our approach that by plugging Y_{n-1} into the formula repeatedly, we can extract π geometrically from the chaotic logistic map.

Convergence to π

Unlike the statistical method, this geometric method converges extremely fast.

n	Y_{-n}	Estimate ($\hat{\pi}$)	Error ($\pi - \hat{\pi}$)
1	5×10^{-1}	2.82842712	3.13×10^{-1}
3	3.8×10^{-2}	3.12144515	2.01×10^{-2}
5	2.4×10^{-3}	3.14033116	1.26×10^{-3}
8	3.8×10^{-5}	3.14157294	1.97×10^{-5}
14	9.2×10^{-9}	3.14159265	1.22×10^{-9}

With just around 10 iterations, we achieve near-perfect precision for π!

Conclusion

We started with asking: Can we find π in chaos?

- The Logistic Map (at $r = 4$) is fully deterministic but chaotic.
- It is structurally isomorphic to a folded sin wave ($\theta \to 2\theta$).

- *Statistically:* We recovered π from the probability density of the chaos.
- *Geometrically:* We recovered π by reversing the chaos to rediscover Vieta's formula.

Chaos is not the absence of order; it is geometry in disguise.

References

[1] Serdar Beji. *"A Geometric Formulation and a Series Approach for Estimating Pi with Remarks on a Sumerian Tablet".* In: *Advances in Pure Mathematics* (2022). DOI: 10.4236/apm.2022.1211045.

[2] Vincent Dumoulin and Félix Thouin. *A Ballistic Monte Carlo Approximation of Pi.* 2014. arXiv: 1404.1499 [physics.pop-ph]. URL: https://arxiv.org/abs/1404.1499.

[3] Frank W. K. Firk. *A proof of the Gregory-Leibniz series and new series for calculating pi.* 2009. arXiv: `0909.1523 [math.HO]`. URL: `https://arxiv.org/abs/0909.1523`.

[4] Milton F. Maritz. *"Extracting Pi from Chaos".* In: *The College Mathematics Journal* 55.2 (2023), pp. 86–99.

[5] François Viète. *Opera Mathematica.* Ed. by Frans van Schooten. Reprint of the 1646 Leiden edition; includes Variorum de Rebus Mathematicis Responsorum (1593). Hildesheim, New York: Georg Olms Verlag, 1970. ISBN: 9783487026022.